

## Quantum deformed magnon kinematics

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**ABSTRACT:** The dispersion relation for planar  $\mathcal{N} = 4$  supersymmetric Yang-Mills is identified with the Casimir of a quantum deformed two-dimensional kinematical symmetry,  $E_q(1,1)$ . The quantum deformed symmetry algebra is generated by the momentum, energy and boost, with deformation parameter  $q = e^{2\pi i/\lambda}$ . Representing the boost as the infinitesimal generator for translations on the rapidity space leads to an elliptic uniformization with crossing transformations implemented through translations by the elliptic half-periods. This quantum deformed algebra can be interpreted as the kinematical symmetry of a discrete integrable model with lattice spacing given by the BMN length  $a = 2\pi/\sqrt{\lambda}$ . The interpretation of the boost generator as the corner transfer matrix is briefly discussed.

**KEYWORDS:** AdS-CFT Correspondence, Quantum Groups.

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## 1. Introduction

An important boost into our current understanding of the AdS/CFT correspondence came from the BMN suggestion to probe sectors with large quantum numbers [1]. The BMN limit provided also an appealing dispersion relation for planar  $\mathcal{N} = 4$  supersymmetric Yang-Mills. The uncovering of integrability both on the gauge [2] and string sides [3] of the correspondence allowed then the search for the explicit form of the scattering matrices of  $\mathcal{N} = 4$  Yang-Mills [4] and of type IIB string theory in  $AdS_5 \times S^5$  [5]. The construction in [4] also implied a derivation in purely algebraic terms of a general dispersion relation of BMN type. This dispersion relation exhibits some sort of double nature, as it looks relativistic in a certain limit, while also includes typical aspects of a lattice dispersion relation. The absence of conventional relativistic invariance is indeed a feature of magnon kinematics in the AdS/CFT correspondence, and requires an elliptic approach to crossing symmetry in the scattering matrix. In [6] an elliptic uniformization was derived and shown to lead to a non-trivial implementation of crossing in terms of translations by half-periods of the elliptic curve defining the kinematical rapidity plane.

In this note we address the problem of the kinematical origin of the BMN type of dispersion relations by identifying the kinematical symmetry group underlying the integrable model. This symmetry is a quantum deformation of the pseudoeuclidean group  $E_q(1, 1)$  [7], with the deformation parameter  $q$  given in terms of the 't Hooft coupling constant by  $q = e^{2\pi i/\lambda}$ . The Casimir of this algebra is indeed the dispersion relation in  $\mathcal{N} = 4$  supersymmetric Yang-Mills, and the boost is the generator of infinitesimal translations on the elliptic rapidity plane. The meaning of this kinematical symmetry must be understood from the structure of the Hopf algebra symmetry [8–10, 5, 11, 12]. In [8] the existence of a central Hopf subalgebra was noticed and the spectrum of this center was proposed as the rapidity plane. This is indeed the usual situation in integrable models of the chiral Potts type, where the kinematical symmetry group acts naturally on the spectrum of the central Hopf subalgebra.

The kinematical symmetry  $E_q(1, 1)$  has non-trivial co-multiplications for the boost generators that are at the root of the elliptic nature of the rapidity space. The co-multiplication rules also underly the non-trivial crossing transformations on the rapidities. Furthermore, as pointed out in [13, 14], these quantum deformed algebras are the natural candidates to kinematical symmetry groups of lattice models, with the lattice spacing being related to the quantum deformation parameter. As we will show in the case of  $\mathcal{N} = 4$  Yang-Mills this lattice spacing can be identified with the scale introduced in [1] through the Yang-Mills interaction between two adjacent points in a BMN operator.

## 2. Elliptic rapidity

We will start by reviewing the uniformization of the Poincaré group in 1+1 dimensions. The energy, momentum and boost generators satisfy

$$[N, P] = E, \quad [N, E] = P, \quad [E, P] = 0. \quad (2.1)$$

If we introduce a rapidity  $z$  in terms of the boost generator through

$$N \equiv \frac{\partial}{\partial z}, \quad (2.2)$$

the algebra (2.1) implies

$$\frac{\partial P(z)}{\partial z} = E(z), \quad (2.3)$$

$$\frac{\partial^2 P(z)}{\partial z^2} = P(z). \quad (2.4)$$

Recalling now the usual relativistic mass shell condition given by the Casimir of the algebra (2.1)

$$E^2 = P^2 + m^2, \quad (2.5)$$

the solution to equations (2.3) and (2.4) are the standard rapidity relations

$$P(z) = m \sinh z, \quad E(z) = m \cosh z. \quad (2.6)$$

Therefore the rapidity  $z$  as defined in (2.2) through the boost generator is the uniformization parameter of the curve (2.5). The universal cover in a standard relativistic theory is the sphere, and thus a trigonometric uniformization is sufficient.

Now let us assume that the commutation relation  $[N, P] = E$  holds and, instead of the standard relativistic dispersion relation, consider the mass shell condition for  $\mathcal{N} = 4$  Yang-Mills [4],

$$E^2 = 1 + \alpha \sin^2 \left( \frac{P}{2} \right), \quad (2.7)$$

where  $\alpha = \lambda/\pi^2$ , with  $\lambda$  the 't Hooft coupling constant. From (2.3) we then get

$$\frac{\partial P(z)}{\partial z} = \sqrt{1 + \alpha \sin^2 \left( \frac{P}{2} \right)}, \quad (2.8)$$

that can be integrated in terms of Jacobi elliptic functions,<sup>1</sup>

$$\sin\left(\frac{P(z)}{2}\right) = \frac{1}{(1+\alpha)^{1/2}} \operatorname{sd}\left(\frac{\alpha^{1/2}z}{2m^{1/2}} \middle| m\right), \quad (2.9)$$

with elliptic modulus  $m^2 \equiv \alpha/(1+\alpha)$ . Thus the rapidity space for the  $\mathcal{N} = 4$  Yang-Mills dispersion relation is a curve of genus one. Let us now consider the relativistic limit of (2.7). This corresponds to the strong-coupling regime, with  $P_{\text{eff}} \equiv \frac{P\alpha^{1/2}}{2}$  finite, so that  $P \ll 1$ . In this limit (2.9) becomes

$$P_{\text{eff}}(z) = \sinh\left(\frac{\alpha^{1/2}z}{2}\right), \quad (2.10)$$

which agrees with (2.6) for an effective relativistic rapidity  $z_{\text{eff}} \equiv z\alpha^{1/2}/2$ .

Once we have determined the elliptic uniformization  $P(z)$  we can easily find out the transformation in the rapidity  $z$  realizing the change under crossing symmetry of the momentum,  $P \rightarrow -P$ . From (2.9) it follows that when  $P \rightarrow -P$  the function  $\operatorname{sd}(\alpha^{1/2}z/2m^{1/2}|m)$  changes sign, which requires shifts by the half-periods  $2K$  and  $2iK'$ , with  $K$  and  $iK'$  the elliptic quarter-periods,

$$K(m) = K'(1-m) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}. \quad (2.11)$$

Therefore the crossing transformation in the rapidity variable  $z$  amounts to

$$z \rightarrow z + \frac{4K}{\sqrt{1+\alpha}}. \quad (2.12)$$

In the relativistic limit defined above the shift (2.12) becomes the standard relativistic crossing transformation.

### 3. The quantum group symmetry

In the previous section we have constructed a rapidity uniformization of the  $\mathcal{N} = 4$  Yang-Mills dispersion relation based on identifying the boost generator with translations in the rapidity plane. However in the derivation we have employed the dispersion relation as an input. A natural question then is what is the kinematical symmetry algebra generated by some  $(1+1)$ -dimensional momentum, energy and boost such that the dispersion relation for  $\mathcal{N} = 4$  Yang-Mills is the corresponding Casimir. Nicely enough this algebra exist and is given by a quantum deformation of the  $(1+1)$ -dimensional pseudoeuclidean algebra, namely  $E_q(1,1)$ , with deformation parameter  $q = e^{ia}$  for  $a$  a real number.

The defining relations of  $E_q(1,1)$  are [7, 13]

$$\begin{aligned} KEK^{-1} &= E, & KNK^{-1} &= N + aE, \\ KK^{-1} &= \mathbb{1}, & NE - EN &= (K - K^{-1})/(2a). \end{aligned} \quad (3.1)$$

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<sup>1</sup>This elliptic uniformization already appeared in [15], and more recently in [16].

By defining  $K \equiv q^{\hat{P}}$  the Casimir for the previous algebra leads to the dispersion relation

$$E^2 = C + \frac{2}{a} \sin^2 \left( \frac{a\hat{P}}{2} \right), \tag{3.2}$$

which is precisely the type of dispersion relation we are looking for, once we perform the identification  $a\hat{P} = P$  and take

$$\alpha = \frac{2}{a}. \tag{3.3}$$

Notice that (3.1) is a quantum deformation of the two-dimensional Poincaré algebra (2.1). Representing now the boost operator as the infinitesimal generator for translations on the rapidity plane leads again to the elliptic uniformization described above. As before the relativistic limit corresponds to  $a = 0$ , with deformation parameter  $q = 1$ .

The co-multiplication rules for the algebra (3.1) are

$$\begin{aligned} \Delta(E) &= K^{-1/2} \otimes E + E \otimes K^{1/2}, \\ \Delta(N) &= K^{-1/2} \otimes N + N \otimes K^{1/2}, \\ \Delta(K) &= K \otimes K, \end{aligned} \tag{3.4}$$

and lead to the following non-trivial antipodes,

$$\begin{aligned} \gamma(E) &= -E, \\ \gamma(N) &= -N - \left( \frac{a}{2} \right) E, \\ \gamma(K) &= K^{-1}. \end{aligned} \tag{3.5}$$

Notice that the antipodes for  $E$  and  $K$  correspond to the crossing transformations. The non-triviality of the antipode for the boost generator already indicates the non-trivial transformation induced by crossing symmetry on the elliptic rapidity plane.

#### 4. The meaning of the boost generator and the quantum deformation parameter

In the previous paragraphs we have identified the kinematic symmetry group for  $\mathcal{N} = 4$  Yang-Mills magnons with the quantum deformed pseudoeuclidean algebra  $E_q(1,1)$ . The Casimir of this algebra leads to the  $\mathcal{N} = 4$  Yang-Mills dispersion relation. Furthermore the representation of the boost generator in terms of translations on the rapidity plane provides the elliptic uniformization. This identification was possible because we have included, in addition to the momentum and the energy, the generator of the boosts. We may now wonder about the meaning of the inclusion of the boost generator for the underlying integrable model. The answer to this question is well known for integrable systems and goes back to Baxter's corner transfer matrix [17]. In fact given the transfer matrix  $T(z)$  for an integrable model the corner transfer matrix generator is simply defined as  $\partial/\partial z$ . This generator, together with the infinite tower of conserved charges (the first two are precisely the momentum and the energy), defines the lattice kinematical group of the integrable

model, with the corresponding rapidity uniformization parameter given by  $z$  [18]. The picture becomes specially clear for the simplest of the chiral Potts models, namely the Ising model, where  $z$  lives on an elliptic curve and where Onsager's uniformization provides the rapidity uniformization for the kinematical symmetry group generated by the corner transfer matrix and the set of conserved charges [18]. The double periodicity of the elliptic functions contains in fact both the symmetry under euclidean rotations and the “Brillouin” periodicity.

In the case we are considering here the quantum deformed algebra  $E_q(1, 1)$  can be interpreted as the kinematical invariance of a discrete system with a lattice spacing together with a continuous time variable. The lattice spacing is determined by the quantum deformation parameter  $a$ . An immediate question is thus what is the meaning of this length scale in the BMN context. In particular in [1] a natural “length” scale was defined as  $a_{\text{BMN}} = 2\pi/\sqrt{\lambda}^2$ . The scale  $a$  obtained in (3.3) turns out to be precisely this BMN length.

An important feature of the algebra that we have identified are the non-trivial co-multiplication rules, together with the antipode for the boost generator. The constraints on the dressing factor imposed by them will be presented elsewhere.

## Acknowledgments

This work is partially supported by the Spanish DGI contract FPA2003-02877 and CAM project HEPHACOS P-ESP-00346.

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<sup>2</sup>See equation A.14 of [1].

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